Local Artin Map via Cohomology

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In this article, we assume K is a non-archimedean local field.

1 Review: Invariant maps for unramified extensions

Recall that for a finite unramified extension of local fields L/K , the invariant map

$$\mathrm{inv}_{L/K} : H^2(\mathrm{Gal}(L/K), L^\times) \to \mathbb{Q}/\mathbb{Z}$$

is defined to be the composition

$$H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow[\cong]{v_L} H^2(\mathrm{Gal}(L/K), \mathbb{Z}) \xleftarrow[\cong]{\delta} H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$
$$f \mapsto f(\mathrm{Frob}_{L/K})$$

where Frob_L/K is the Frobenius element for L/K. Notice that the invariant map is injective and functorial in L: for $K \subset L \subset M$ finite unramified extensions, the following diagram commutes

$$\begin{array}{ccc} H^2(\mathrm{Gal}(L/K), L^\times) & \xrightarrow{\mathrm{inv}_{L/K}} & \mathbb{Q}/\mathbb{Z} \\ \Big\downarrow \mathrm{Inf} & \nearrow_{\mathrm{inv}_{M/K}} & \\ H^2(\mathrm{Gal}(M/K), M^\times) & & \end{array}.$$

Theorem 1.1. *Let K be a non-archimedean local field. There exists an isomorphism*

$$inv_K : H^2(Gal(K^{ur}/K), (K^{ur})^\times) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

such that the composition with the inflation map induces an isomorphism

$$inv_{L/K} : H^2(Gal(L/K), L^\times) \xrightarrow{\cong} \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

for every finite unramified extension L/K.

2 Invariant maps for K^{sep}/K

In this section we will discuss the extension of the invariant map to a separable closure K^{sep} of a non-archimedean local field K.

Proposition 2.1. *Let L/K be a finite separable extension of non-archimedean local fields. Then, there exists a homomorphism*

$$\phi : H^2(Gal(K^{ur}/K), (K^{ur})^\times) \to H^2(Gal(L^{ur}/L), (L^{ur})^\times)$$

such that the diagram commutes

$$\begin{array}{ccc} H^2(Gal(K^{ur}/K), (K^{ur})^\times) & \xrightarrow[\cong]{inv_K} & \mathbb{Q}/\mathbb{Z} \\ \downarrow \phi & & \downarrow [L:K]\cdot \\ H^2(Gal(L^{ur}/L), (L^{ur})^\times) & \xrightarrow[\cong]{inv_L} & \mathbb{Q}/\mathbb{Z} \end{array}$$

Furthermore, if L/K is Galois, then the kernel $ker(\phi)$ can be identified as a cyclic subgroup of order $[L:K]$ in $H^2(Gal(L/K), L^\times)$.

Proof. First suppose L/K is a finite Galois extension. The Hilbert's 90 tells us that $H^1(\text{Gal}(L^{ur}/L), (L^{ur})^\times) = 0$ and $H^1(\text{Gal}(L^{ur}/K^{ur}), (L^{ur})^\times) = 0$. Therefore, we have the following inflation-restriction sequences:

$$0 \to H^2(\text{Gal}(L/K), L^\times) \xrightarrow{\text{Inf}} H^2(\text{Gal}(L^{ur}/K), (L^{ur})^\times) \xrightarrow{\text{Res}} H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times)$$
$$0 \to H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) \xrightarrow{\text{Inf'}} H^2(\text{Gal}(L^{ur}/K), (L^{ur})^\times) \xrightarrow{\text{Res'}} H^2(\text{Gal}(L^{ur}/K^{ur}), (L^{ur})^\times).$$

Define $\phi : H^2(Gal(K^{ur}/K), (K^{ur})^\times) \to H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times)$ as

$$\phi = \text{Res} \circ \text{Inf}'.$$

Now, assuming ϕ is the map in the first part of the proposition, we then have

$$\ker(\phi) \cong \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

under the invariant map. Note that

$$Inf'(\ker(\phi)) \subseteq \ker(\text{Res}) = \text{Im}(Inf) = H^2(\text{Gal}(L/K), L^\times)$$

and so $\ker(\phi)$ can be identified as a cylic subgroup of oerder $[L:K]$ in $H^2(\text{Gal}(L/K), L^\times)$.

Now, we drop the Galois assumption and prove the first part of the proposition. Notice that the map $\phi = \text{Res} \circ \text{Inf}'$ still makes sense even when L/K is not Galois. Recall that $[L:K] = ef$ where e is the ramification index and f is the inertia degree.

Writing out the definitions of inv_K and inv_L:

$$\begin{array}{ccccccc} H^2(\text{Gal}(K^{ur}/K), (K^{ur})^\times) & \xrightarrow[\cong]{v_K} & H^2(\text{Gal}(K^{ur}/K), \mathbb{Z}) & \xleftarrow[\cong]{\delta} & H^1(\text{Gal}(K^{ur}/K), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{g \mapsto g(\text{Frob}_K)} & \mathbb{Q}/\mathbb{Z} \\ \downarrow \phi & & \downarrow e\phi & & \downarrow e\phi & & \downarrow n=ef\cdot \\ H^2(\text{Gal}(L^{ur}/L), (L^{ur})^\times) & \xrightarrow[\cong]{v_L} & H^2(\text{Gal}(L^{ur}/L), \mathbb{Z}) & \xleftarrow[\cong]{\delta} & H^1(\text{Gal}(L^{ur}/L), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{g \mapsto g(\text{Frob}_L)} & \mathbb{Q}/\mathbb{Z} \end{array}$$

The first commutative square is given by the valuations and the fact that $v_L|_{K^{ur}} = ev_K$.

$$\begin{array}{ccc} (K^{ur})^\times & \xrightarrow{v_K} & \mathbb{Z} \\ \big\downarrow & & \big\downarrow e\,\cdot \\ (L^{ur})^\times & \xrightarrow{v_L} & \mathbb{Z} \end{array}$$

For the second commutative square, we have used the fact that ϕ is defined by the composition of inflation and restriction maps, both of which commute with the boundary maps.

For the last commutative square, note that by writing $L^{ur} = L \cdot K^{ur}$, we have an injection

$$\begin{gathered} \mathrm{Gal}(L^{ur}/L) \hookrightarrow \mathrm{Gal}(K^{ur}/K) \\ \sigma \mapsto \sigma|_{K^{ur}}. \end{gathered}$$

In particular,

$$\mathrm{Frob}_L|_{K^{ur}} = \mathrm{Frob}_K^f.$$

So, for $g \in H^1(\mathrm{Gal}(K^{ur}/K), \mathbb{Q}/\mathbb{Z})$,

$$g|_{\mathrm{Gal}(L^{ur}/L)}(\mathrm{Frob}_L) = g(\mathrm{Frob}_K^f) = f \cdot g(\mathrm{Frob}_K).$$

□

Now, we want to extend inv_K to arbitrary separable extensions of K. For this we require what Neukirch calls the class field axiom:

Theorem 2.2. *(**Class Field Axiom**) Let L/K be a cyclic extension of non-archimedean local fields. Then,*

$$\left|H^k(Gal(L/K), L^\times)\right| = \begin{cases} [L:K], & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

Corollary 2.3. *Let L/K be a finite Galois extensiono of non-archimedean local fields. Then, $H^2(Gal(L/K), L^\times)$ is cyclic of order $[L:K]$.*

Proof. We know from proposition 2.1 that $H^2(\mathrm{Gal}(L/K), L^\times)$ contains a cyclic subgroup of order $[L:K]$ and so we need to show they are equal.

We prove this by induction on $[L:K]$. First note that if L/K is cyclic, then the result follows from the class field axiom. Also, it is a fact from number theory that the Galois group of an extension of non-archimedean local fields is solvable. So by Galois correspondence, there exists a tower of $K \subsetneq E \subsetneq L$ of Galois extensions with

$$\begin{gathered} \left|H^2(\mathrm{Gal}(L/E), L^\times)\right| = [L:E] \\ \left|H^2(\mathrm{Gal}(E/K), E^\times)\right| = [E:K]. \end{gathered}$$

On the other hand, we have the inflation-restriction sequence

$$0 \to H^2(\mathrm{Gal}(E/K), E^\times) \xrightarrow{\mathrm{Inf}} H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\mathrm{Res}} H^2(\mathrm{Gal}(L/E), L^\times).$$

It follows that

$$\left|H^2(\mathrm{Gal}(L/K),L^\times)\right| \le \left|H^2(\mathrm{Gal}(L/E),L^\times)\right| \cdot \left|H^2(\mathrm{Gal}(E/K),E^\times)\right| = [L:K].$$

□

Now, we show that the invariant map extends to a separable closure K^{sep}.

Theorem 2.4. *Let K be a non-archimedean local field. There is an isomorphism*

$$inv_K : H^2(Gal(K^{sep}/K),(K^{sep})^\times) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

such that for every finite Galois extension L/K, the composition with the inflation map induces

$$inv_{L/K} : H^2(Gal(L/K),L^\times) \xrightarrow{\cong} \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$$

which coincides with the invariant map when L/K is unramified. Moreover,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(Gal(L/K),L^\times) & \xrightarrow{Inf} & H^2(Gal(K^{sep}/K),(K^{sep})^\times) & \xrightarrow{Res} & H^2(Gal(K^{sep}/L),(K^{sep})^\times) & \longrightarrow & 0 \\ & & \cong\downarrow inv_{L/K} & & \cong\downarrow inv_K & & \cong\downarrow inv_L & & \\ 0 & \longrightarrow & \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \xrightarrow{[L:K]} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0. \end{array}$$

Proof. As a consequence of Hilbert's 90, the inflation map

$$\mathrm{Inf} : H^2(\mathrm{Gal}(K^{ur}/K),(K^{ur})^\times) \to H^2(\mathrm{Gal}(K^{sep}/K,(K^{sep})^\times)$$

is injective. From proposition 2.1 and corollary 2.3, for all finite Galois extensions L/K, under the homomorphism

$$\phi : H^2(\mathrm{Gal}(K^{ur}/K),(K^{ur})^\times) \to H^2(\mathrm{Gal}(L^{ur}/L),(L^{ur})^\times),$$

the kernel $\ker(\phi) = H^2(\mathrm{Gal}(L/K),L^\times)$ is a cyclic subgroup of order $[L:K]$ in $H^2(\mathrm{Gal}(K^{ur}/K),(K^{ur})^\times)$. But since the cohomology group is the direct system of the system formed by the inflation maps between finite Galois extensions of K, that is

$$H^2(\mathrm{Gal}(K^{sep}/K),(K^{sep})^\times) = \varinjlim_{L/K \text{finite Galois}} H^2(\mathrm{Gal}(L/K),L^\times),$$

the inflation map $\mathrm{Inf} : H^2(\mathrm{Gal}(K^{ur}/K),(K^{ur})^\times) \to H^2(\mathrm{Gal}(K^{sep}/K,(K^{sep})^\times)$ is an isomorphism. □

3 The fundamental class

Definition 3.1. Let L/K be a finite Galois extension of non-archimedean local fields. The *fundamental class* $u_{L/K} \in H^2(\mathrm{Gal}(L/K), L^\times)$ is the preimage of $\frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$ under the invariant map $\mathrm{inv}_{L/K}$.

We have some nice properties of the fundamental classes in a tower of Galois extensions.

Lemma 3.2. *Suppose $K \subseteq E \subseteq L$ is a tower of finite Galois extensions of non-archimedean local fields. Then,*

(a) *Res*$(u_{L/K}) = u_{L/E}$.

(b) *CoRes*$(u_{L/E}) = [E : K]u_{L/K}$.

(c) *Inf*$(u_{E/K}) = [L : E]u_{L/K}$.

Proof. For part (c), recall that the inflation map Inf commutes with the invariant maps, that is

$$\begin{array}{ccc} H^2(\mathrm{Gal}(E/K), E^\times) & \xrightarrow{\mathrm{inv}_{E/K}} & \mathbb{Q}/\mathbb{Z} \\ \big\downarrow \mathrm{Inf} & \nearrow_{\mathrm{inv}_{L/K}} & \\ H^2(\mathrm{Gal}(L/K), L^\times) & & \end{array} .$$

Since $\mathrm{inv}_{E/K}(u_{E/K}) = \frac{1}{[E:K]}\mathbb{Z}/\mathbb{Z}$ and $\mathrm{inv}_{L/K} = \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z}$, we have

$$\mathrm{Inf}(u_{E/K}) = [L : E]u_{L/K}.$$

For part (a), we claim that

$$\begin{array}{ccccc} H^2(\mathrm{Gal}(L/K), L^\times) & \xhookrightarrow{\mathrm{Inf}} & H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^\times) & \xrightarrow{\mathrm{inv}_K} & \mathbb{Q}/\mathbb{Z}a \\ \big\downarrow \mathrm{Res} & & \big\downarrow \mathrm{Res} & & \big\downarrow [\mathrm{E{:}K}] \\ H^2(\mathrm{Gal}(L/E), L^\times) & \xhookrightarrow{\mathrm{Inf}} & H^2(\mathrm{Gal}(K^{sep}/E), (K^{sep})^\times) & \xrightarrow{\mathrm{inv}_E} & \mathbb{Q}/\mathbb{Z} \end{array} .$$

The second commutative square is given by theorem 2.4. We check that the induced map on $H^2(\mathrm{Gal}(L/K), L^\times)$ is the restriction map into $H^2(\mathrm{Gal}(L/E), L^\times)$. Notice the map the restriction map Res : $H^2(\mathrm{Gal}(K^{sep}/K), (K^{sep})^\times)$ is induced by the inclusion $\mathrm{Gal}(K^{sep}/E) \hookrightarrow \mathrm{Gal}(K^{sep}/E)$ whereas the inflation maps are given by quotient maps $\mathrm{Gal}(K^{sep}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$ and $\mathrm{Gal}(K^{sep}/E) \twoheadrightarrow \mathrm{Gal}(L/E)$:

$$\begin{array}{ccc} \mathrm{Gal}(L/K) = \mathrm{Gal}(K^{sep}/K) \big/ \mathrm{Gal}(K^{sep}/L) & \twoheadleftarrow & \mathrm{Gal}(K^{sep}/K) \\ \big\uparrow i & & \big\uparrow \\ \mathrm{Gal}(L/E) = \mathrm{Gal}(K^{sep}/E) \big/ \mathrm{Gal}(K^{sep}/L) & \twoheadleftarrow & \mathrm{Gal}(K^{sep}/E) \end{array}$$

and hence the map i is an inclusion. This shows that the induced map on $H^2(\mathrm{Gal}(L/K), L^\times)$ is the restriction map into $H^2(\mathrm{Gal}(L/E), L^\times)$, which proves part (a).

For part (b), we have

$$\mathrm{CoRes}(u_{L/E}) = \mathrm{CoRes}(Res(u_{L/K})) = [E : K]u_{L/K}.$$

□

4 The Local Artin Map for K^{ab}/K

Our ultimate goal in this section is to define the local Artin map for K^{ab}/K. First, we define the local Artin map for a finite Galois extension L/K of non-archimedean local fields. Recall that for $G = \mathrm{Gal}(L/K)$, we have the isomorphisms

$$G^{ab} \xrightarrow[\cong]{g \mapsto [g-1]} I_G/I_G^2 \xleftarrow[\cong]{\delta_0} H_1(G, \mathbb{Z}) =: \widehat{H}^{-2}(G, \mathbb{Z})$$

where the second isomorphism is given by the short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Furthermore, for any subgroup $H \subset G = \mathrm{Gal}(L/K)$, we have $H^1(H, L^\times) = 0$ and $H^2(H, L^\times)$ is cyclic of order $|H|$. Hence, Tate's theorem gives an isomorpshism

$$\widehat{H}^n(G, \mathbb{Z}) \xrightarrow[\cong]{\Phi_{L/K}} \widehat{H}^{n+2}(G, L^\times)$$

for a choice of generator of $\widehat{H}^2(G, L^\times)$.

Definition 4.1. Let L/K be a finite Galois extension of non-archimedean local fields with Galois group $\mathrm{Gal}(L/K)$. The *local Artin map*

$$\theta_{L/K} : K^\times/N_{L/K}(L^\times) \xrightarrow{\cong} \mathrm{Gal}(L/K)^{ab}$$

is the inverse of the isomorphisms

$$\mathrm{Gal}(L/K)^{ab} \cong \widehat{H}^{-2}(\mathrm{Gal}(L/K), \mathbb{Z}) \xrightarrow[\cong]{\Phi_{L/K}} \widehat{H}^0(\mathrm{Gal}(L/K), L^\times) =: K^\times/N_{L/K}(L^\times)$$

where $\Phi_{L/K}$ is given by Tate's theorem when we take the fundamental class $u_{L/K}$ as our generator of $H^2(\mathrm{Gal}(L/K), L^\times)$.

We will also view $\theta_{L/K} : K^\times \to \mathrm{Gal}(L/K)^{ab}$ as a surjective map with kernel $N_{L/K}(L^\times)$.

Lemma 4.2. *Let $K \subseteq E \subseteq L$ be a tower of finite Galois extensions of non-archimedean local fields. Then,*

$$\begin{array}{ccc} E^\times & \xrightarrow{\theta_{L/E}} & Gal(L/E)^{ab} \\ \downarrow{\scriptstyle N_{E/K}} & & \downarrow \\ K^\times & \xrightarrow{\theta_{L/K}} & Gal(L/K)^{ab} \end{array}.$$

Proof. Let $G := \mathrm{Gal}(L/K)$ and $H := \mathrm{Gal}(L/E)$. Hence, $\mathrm{Gal}(E/K) = G/H$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} E^\times & \longrightarrow & E^\times/N_{L/E}(L^\times) & = & \widehat{H}^0(H, L^\times) & \xrightarrow[\cong]{\Phi_{E/K}^{-1}} & \widehat{H}^2(H, \mathbb{Z}) & \xrightarrow{\cong} & H^{ab} \\ \downarrow{\scriptstyle N_{E/K}} & & \downarrow{\scriptstyle N_{E/K}} & & {\scriptstyle \mathrm{CoRes}=N_{G/H}}\downarrow & & \downarrow{\scriptstyle \mathrm{CoRes}} & & \downarrow \\ K^\times & \longrightarrow & K^\times/N_{L/K}(L^\times) & = & \widehat{H}^0(G, L^\times) & \xrightarrow[\cong]{\Phi_{L/K}^{-1}} & \widehat{H}^2(G, \mathbb{Z}) & \xrightarrow{\cong} & G^{ab} \end{array}$$

□

An alternative way to describe the local Artin map is via cup products (see [1]). For our purposes, we only need to look at the cup products on the 0-th and the 2-nd degree. For a Galois extension L/K of non-archimedean local fields, we have

$$\begin{aligned} H^0(\mathrm{Gal}(L/K), L^\times) \times H^2(\mathrm{Gal}(L/K), \mathbb{Z}) &\xrightarrow{\cup} H^2(\mathrm{Gal}(L/K), L^\times) \\ (a, f) &\mapsto a \cup f \end{aligned}$$

such that

$$(a \cup f)(g_1, g_2) = a \otimes_{\mathbb{Z}} f(g_1, g_2) = a^{f(g_1, g_2)}.$$

To get the last equality, note that the $\mathbb{Z}$-action on $K^\times$ is defined as follows: for $z \in \mathbb{Z}$, $a \in K^\times$,

$$z \cdot a = a^z.$$

Lemma 4.3. *For every $\chi \in Hom(Gal(L/K), \mathbb{Q}/\mathbb{Z}) = H^1(Gal(L/K), \mathbb{Q}/\mathbb{Z})$ and $a \in K^\times$,*

$$\chi(\theta_{L/K}(a)) = inv_{L/K}(a \cup \delta\chi)$$

where δ is the boundary map

$$\delta : H^1(Gal(L/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^2(Gal(L/K), \mathbb{Z}).$$

In order to define the local Artin map from $K^\times$ to $\mathrm{Gal}(K^{ab}/K)$, we need the following lemma:

Lemma 4.4. *Suppose $K \subseteq E \subseteq L$ is a tower of finite Galois extensions of non-archimedean local fields. Then,*

$$\begin{array}{ccc} K^\times & \xrightarrow{\theta_{L/K}} & Gal(L/K) \\ & \overset{\theta_{E/K}}{\searrow} & \downarrow \\ & & Gal(E/K) \end{array}.$$

In other words,

$$\theta_{L/K}(a)|_E = \theta_{E/K}(a)$$

for all $a \in K^\times$.

Proof. Notice that any homomorphism $\chi : \mathrm{Gal}(E/K) \to \mathbb{Q}/\mathbb{Z}$ extends to a map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(E/K) \to \mathbb{Q}/\mathbb{Z}$ which will still be called χ. Now to prove the lemma, it is enough to prove that

$$\chi(\theta_{L/K}(a)) = \chi(\theta_{E/K}(a))$$

for all $\chi : \mathrm{Gal}(E/K) \to \mathbb{Q}/\mathbb{Z}$ and all $a \in K^\times$. But then by using the formula in previous lemma, it suffices to prove that

$$\mathrm{inv}_{L/K}(a \cup \delta_{L/K}\chi) = \mathrm{inv}_{E/K}(a \cup \delta_{E/K}\chi).$$

But this follows from $\mathrm{inv}_{E/K} = \mathrm{inv}_{L/K} \circ \mathrm{Inf}$ and the fact that inflation map commutes with cup product. □

Now we can extend the local Artin map to a maximal abelian extension K^{ab}:

Definition 4.5. The *local Artin map* for a local field K

$$\theta_K : K^\times \to \mathrm{Gal}(K^{ab}/K)$$

is a continuous group homomorphism defined by a compatible system of maps $\{\theta_{L/K}\}$.

The following theorem tells us the action of $K^\times$ on the maximal unramified extension K^{ur} in K^{ab}.

Theorem 4.6. *Let $Frob_K$ be the Frobenius element of K and v_K be the discrete valuation on $K^\times$. Then,*

$$\theta_K(a)|_{K^{ur}} = Frob_K^{v_K(a)}$$

for all $a \in K^\times$. In particular, $\theta_K(\pi) = Frob_K$ for any uniformizer π of K.

Proof. By the definition of local Artin map, it is sufficient to show that

$$\theta_{L/K}(a) = \mathrm{Frob}_{L/K}^{v_K(a)}$$

for every finite unramofied extension L/K.

Writing out the definition of the invariant map $\mathrm{inv}_{L/K}$, we have

$$H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow[\cong]{v_L} H^2(\mathrm{Gal}(L/K), \mathbb{Z}) \xrightarrow[\cong]{\delta^{-1}} H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{g \mapsto g(\mathrm{Frob}_{L/K})} \mathbb{Q}/\mathbb{Z} .$$

Let $a \in K^\times$ and $\chi : \mathrm{Gal}(L/K) \to \mathbb{Q}/\mathbb{Z}$ be any homomorphism. Then under the maps,

$$a \cup \delta\chi \mapsto v_L(a) \cup \delta\chi \mapsto v_L(a)\chi \mapsto \chi(\mathrm{Frob}_{L/K}^{v_L(a)}) = \chi(\mathrm{Frob}_{L/K}^{v_K(a)})$$

and hence $\mathrm{inv}_{L/K}(a \cup \delta\chi) = \chi(\mathrm{Frob}_{L/K}^{v_K(a)})$. We can verify this in details: for the first arrow, given $g_1, g_2 \in \mathrm{Gal}(L/K)$,

$$v_L(a \cup \delta\chi)(g_1, g_2) = v_L((a \cup \delta\chi)(g_1, g_2)) = v_L(a \otimes_{\mathbb{Z}} \delta\chi(g_1, g_2)) = v_L(a^{\delta\chi(g_1,g_2)}) = \delta\chi(g_1, g_2)v_L(a)$$
$$(v_L(a) \cup \delta\chi)(g_1, g_2) = v_L(a) \otimes_{\mathbb{Z}} \delta\chi(g_1, g_2) = v_L(a)\delta\chi(g_1, g_2)$$

and so $v_L(a \cup \delta\chi) = v_L(a) \cup \delta\chi$. For the second map, note that

$$\delta(v_L(a)\chi) = \delta(v_L(a)) \cup \chi + (-1)^0 v_L(a) \cup \delta\chi = v_L(a) \cup \delta\chi$$

and so $\delta^{-1}(v_L(a) \cup \delta\chi) = v_L(a)\chi$. The last map is just evaluation map.

On the other hand, by using the formula in lemma 4.3,we have

$$\chi(\theta_{L/K}(a)) = \mathrm{inv}_{L/K}(a \cup \delta\chi) = \chi(\mathrm{Frob}_{L/K}^{v_K(a)})$$

for all $\chi \in H^1(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z})$. It follows that

$$\theta_{L/K}(a) = \mathrm{Frob}_{L/K}^{v_K(a)}.$$

□

We conclude the section with a theorem that says there is no hope to extend the local Artin reciprocity theorems to non-abelian extensions.

Theorem 4.7. *(**Norm Limitation Theorem**) Suppose L/K is a finite separable extension of non-archimedean local fields and E/K is the maximal abelian subextension in L. Then,*

$$N_{L/K}(L^\times) = N_{E/K}(E^\times).$$

Proof. By the transitivity of the norm map, $N_{L/K} = N_{E/K} \circ N_{L/E}$, we see that $N_{L/K}(L^\times) \subseteq N_{E/K}(E^\times)$.

We first assume L/K is Galois. Then the local Artin maps for L/K and E/K are given by

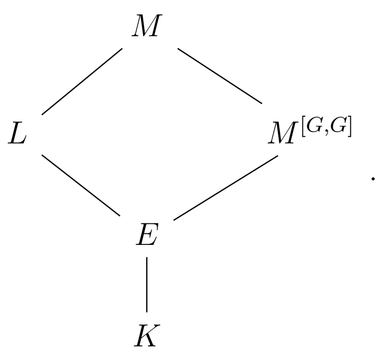
$$\theta_{L/K} : K^\times / N_{L/K}(L^\times) \xrightarrow{\cong} \mathrm{Gal}(L/K)^{ab} = \mathrm{Gal}(E/K)$$
$$\theta_{E/K} : K^\times / N_{E/K}(E^\times) \xrightarrow{\cong} \mathrm{Gal}(E/K)^{ab} = \mathrm{Gal}(E/K).$$

This shows that $N_{L/K}(L^\times) = N_{E/K}(E^\times)$.

For a general case, let M/K be the Galois closure of L. Denote $G := \mathrm{Gal}(M/K)$ and $H := \mathrm{Gal}(M/L)$. We claim that

$$[G, G]H = \mathrm{Gal}(M/E).$$

In fact we have the tower of Galois extensions:

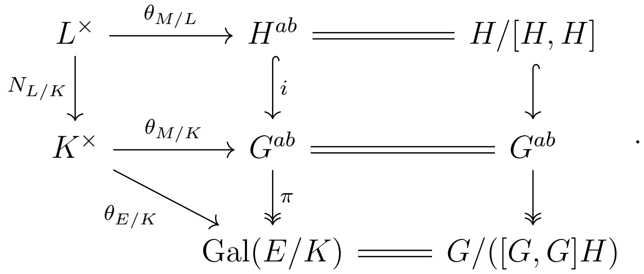


By assumption, E is the maximal abelian extension in L. On the other hand, $M^{[G,G]}$ is the maximal abelian extension in M, so we must have

$$E = L \cap M^{[G,G]} = M^H \cap M^{[G,G]} = M^{[G,G]H}$$

and so by Galois correspondence $[G, G]H = \mathrm{Gal}(M/E)$.

Consider the following diagram:



We want to show that $N_{E/K}(E^\times) \subseteq N_{L/K}(L^\times)$. Take $a \in N_{E/K}(E^\times) = \ker(\theta_{E/K}) = \ker(\pi \circ \theta_{M/K})$. Then we have

$$\theta_{M/K}(a) \in \ker(\pi) = \mathrm{Im}(i).$$

So, there exists $b \in L^\times$ such that

$$\theta_{M/K}(a) = \theta_{M/L}(b) = \theta_{M/K}(N_{L/K}(b)).$$

But this means

$$a \in N_{L/K}(b)\ker(\theta_{M/K}) = N_{L/K}(b)N_{M/K}(M^\times) = N_{L/K}(b) \cdot N_{L/K}(N_{M/L}(M^\times)) \subset N_{L/K}(L^\times)$$

as desired. □

References

[1] Romyar Sharifi, *Group and Galois Cohomology.* `http://math.ucla.edu/~sharifi/lecnotes.html`.

[2] J.S. Milne, *Class Field Theory*, Unpublished, 2013. `http://www.jmilne.org/math/CourseNotes/`